# $\alpha$ emitters have a huge range of half-lives but all $\alpha$s have about the same energy. Why?

The half-lives of alpha-particle emitting radionuclides range from $8 \times 10^{-17}$ seconds to trillions of years. However, the observed *energies* of the emitted particles only range from about 3-10 MeV. Why? We consider elementary nuclear physics to find the answers.

## Half-lives

For a classical 1-dimensional system with energy $E = \frac{p^2}{2m} + V(x)$, to lowest order in $\hbar$ the wavefunction can be recast as

$$\psi(x) \simeq \psi_{WKB}(x) = \frac{C}{\sqrt{p(x)}} \exp\left[\frac{i}{\hbar}\int^x dx'\ p(x')\right]$$

In keeping with the interpretation of $|\psi|^2$, the probability to find the quantum particle in a range $x \rightarrow x + dx$ is inversely proportional to its speed. The slower the particle moves, the more likely (classically) we are to find it in the range $dx$. Evidently this prefactor diverges where $V(x) = E$, the classical 'turning points'.

For tunneling, $p^2(x) < 0$ and an exponentially growing solution is forbidden, so $p(x) = i\sqrt{2m\,[V(x) - E]}$ and the probability to find the particle to have tunneled through a barrier for a strictly radial problem is

$$P_{tun}(r_>) \simeq \frac{C'}{|p(r)|} \exp\left(-2\sqrt{\frac{2m}{\hbar^2}} \int_R^{r_>} dr' \sqrt{V(r) - E}\right)$$

where the barrier is defined by the region $R \leq r \leq r_>$, the classically forbidden zone for energy $E$.

We now consider pre-existing alpha particles confined within the schematic nuclear potential shown in Fig. 1. Having tunneled through the barrier shown, the alpha particle experiences an ordinary Coulomb repulsion from the nucleus left behind [originally of charge $Ze$, now of charge $(Z-2)e$]. Far from the nucleus this potential is zero and the energy $E$ is strictly kinetic energy, which we will now denote as $Q$, a measurable quantity.

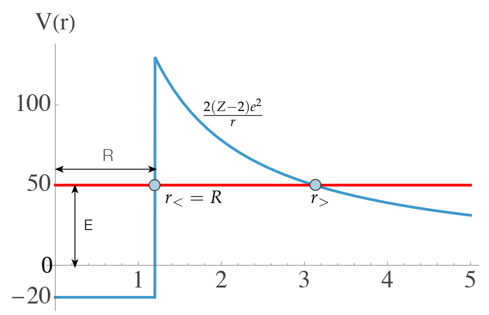


Figure 1: Schematic potential out of which an alpha particle must tunnel to be found infinitely far away.

Alpha particles are known to exist [1] within nuclei and often determine details of light nuclei. 'Even-even' nuclei (consisting of an even number of protons and of neutrons) are especially stable and may be regarded as actually *composed* of alpha particles. See Scholarpedia.

Provided the range of $r$ integration is much larger than the region over which $|p(r')|$ is near zero[1] we may entirely neglect $C'/\,|p(r)|$ and the approximate transmission coefficient is

$$T_{barr} \simeq \exp\left(-2\sqrt{\frac{2(Z-2)}{a_0}\frac{m_\alpha}{m_{el}}} \int_R^{r_>} dr \sqrt{\frac{1}{r} - \frac{1}{r_>}}\right) \tag{1}$$

[1] Otherwise the full WKB 'connection formulae' are required.

where we have used the alpha particle mass $m_\alpha$ and the useful fact that in cgs units $e^2/\hbar^2 = 1/(m_{el}\,a_0)$ with $m_{el}$ the electron mass and $a_0$ the Bohr radius. We have replaced the energy $E = Q$ by the value of the Coulomb potential at $r_>$ (see the diagram), $2(Z-2)e^2/r_>$, yielding the simplest integrand. *Sanity check:* if the barrier shrinks to zero spatial extent, this expression becomes 1, as one expects for the

The integral itself is

$$\sqrt{R}\left\{\sqrt{x}\tan^{-1}\sqrt{x-1} - \sqrt{1-1/x}\right\}$$
$$\simeq \sqrt{R}\left(-2 + \frac{\pi\sqrt{x}}{2} + \frac{1}{3x} + \dots\right) \tag{2}$$

for large $x$, where $x = r_>/R$. Other forms may be found that are equivalent.

transmission coefficient through an arbitrarily thin barrier (that is, no barrier). Thus, approximate as it is, Eq. 1 is expected to be semiquantitatively reliable *provided* the barrier thickness is much larger than the range over which $|p(r')|$ is near zero. With the integral evaluated [Eq. 2] we may replace $r_>$ by $2(Z-2)e^2/Q$.

The radial integral depends on the potential *outside* the nucleus, hence not on the depth of the nuclear well. We can compare $r_>$ with the Bohr radius $a_0 \simeq 0.5292 \times 10^{-8}$cm via

$$\frac{r_>}{a_0} = 2\frac{(Z-2)e^2}{Q\, a_0} \simeq \frac{Z\ 54.4 \text{ eV}}{5 \text{ MeV}} \simeq 9.8 \times 10^{-4} \tag{3}$$

since $e^2/a_0 = 27.21$ eV. We have used an estimate of 5 MeV for $Q$ and $Z = 90$. Using the approximate nuclear radius $R \simeq 1.2A^{1/3}$ fm (fermis or femtometers) for atomic weight $A$ we find, in cm

$$R \simeq 6.2 \times 10^{-13}[A = 240] \tag{4}$$

$$r_> \simeq 5.18 \times 10^{-12}[Z = 90, Q = 5 \text{ MeV}] \tag{5}$$

Thus the estimated $x \simeq 8$ suggests that $x = r_>/R$ is appreciably larger than 1, suggesting that the first two terms in the series suffice for semiquantitative agreement. A comparison in Fig. 2 shows that in fact this is an excellent approximation for moderate to large $x$.

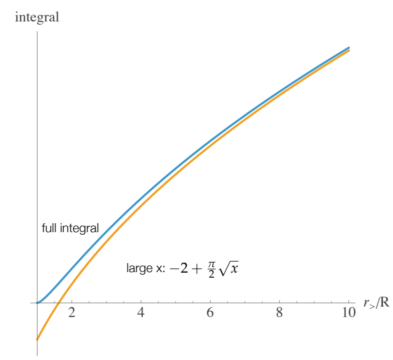


Figure 2: The exact and large-x series expansion of the radial integral as it depends on $x = r_>/R$.

In terms of constants we may now write

$$T_{barr} \simeq \exp\left\{-2\pi(Z-2)\sqrt{\frac{m_\alpha}{m_{el}}} \times \frac{e}{\sqrt{Qa_0}}\right\} \tag{6}$$

provided $x \gg 1$, which in turn requires

$$(Z-2) \gg 0.3472 \times Q[\text{MeV}] \times R[\text{fm}] \tag{7}$$

Each time an alpha particle tunnels through the barrier and escapes, the original nucleus has undergone alpha decay. Thus the tunneling *rate* (or decays per second per nucleus) determines the half-life of the radioisotope. One commonly writes the tunneling rate $r$ as

$$r = \nu T_{barr} \tag{8}$$

where $\nu$ is the 'attempt frequency', the number of times (per second) an alpha particle inside the nucleus encounters the potential barrier. This is reasonably estimated as $\nu = v_\alpha/R$, where $R$ is the nuclear radius and $v_\alpha$ is the average speed of an alpha particle within the nucleus. From this, $\nu \simeq 8.8 \times 10^{21}$ attempts/sec. We will see that the barrier penetration factor $T_{barr}$ can be prodigiously small.

Using a kinetic energy of 5 MeV for such an incident alpha particle and $R \simeq 1.2A^{\frac{1}{3}}$ for $A = 100$, $v_\alpha \simeq 4.9 \times 10^9$ cm/sec.

The decay of a collection of identical radionuclides is exponential:

$$N(t) = N_0\, e^{-r\,t} = N_0\, 2^{-\frac{t}{T_{\frac{1}{2}}}} \quad \text{where } T_{\frac{1}{2}} = \frac{\ln 2}{r} = \frac{\ln 2}{\nu T_{barr}} \tag{9}$$

where $r$ is given in Eq. 8 and $N_0$ is the initial number of nuceli. Thus, finally

$$\log_{10} T_{\frac{1}{2}} \simeq a + b\frac{Z-2}{\sqrt{Q}} \tag{10}$$

where

$$a = \log_{10}\frac{\ln 2}{\nu} \quad b = \frac{2\pi}{\ln 10}\sqrt{\frac{m_\alpha}{m_{el}}}\sqrt{\frac{e^2}{a_0}} \tag{11}$$

The form of Eq. 10 is known as the Geiger-Nuttall law.

The parameter $a$ depends weakly on $Q$ and $A$. Using a reasonable value and the natural constants, we find

$$\log_{10} T_{\frac{1}{2}} \simeq -21.7 + 1.21573\frac{(Z-2)}{\sqrt{Q}} \tag{12}$$

This is essentially a 'first principles' estimate of half-lives agains alpha particle decay across the periodic table.

A fit to a large number of alpha emitters [panel (a) of Fig. 3 exhibits a great deal of scatter about the expected line, simply because *none* of the 'shell' structure of energy levels of nucleons in a nucleus was included. When particular isotopes of a particular element are plotted, it is clear that the agreemnent with the derived form is very good, with the slopes of the fit lines increasing with $Z$ as expected.

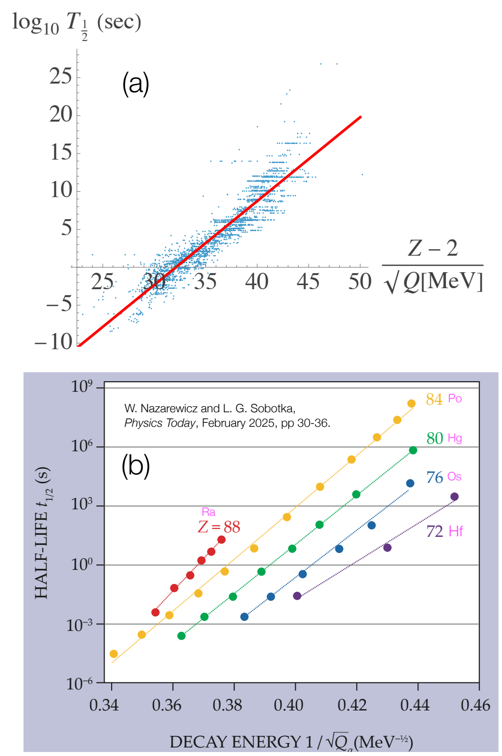


Figure 3: Measured alpha-emitter half-lives plotted according to the Geiger-Nuttall form. Panel (b) is used with permission of the authors of the article cited.

## *Energies of alpha particles*

The Geiger-Nuttall description correctly identified that tunnelling through the Coulomb barrier is the source of trends in the half-life of alpha particle emitters. However, remains unclear why there is a relatively narrow range of alpha kinetice energies $Q$ (roughly 3-10 MeV).

The rest energy of a proton is 938 MeV and for a neutron is 939 MeV. These far exceed the excitations within a nucleus (less than tens of MeV), so nucleons may reasonably be treated as non-relativistic. In the schematic diagram and in computations of the tunneling rate, $E$ was regarded as a continuous variable.

The strength of the *strong force* is such the Coulomb repulsion among protons in the nucleus is a small perturbation. Neglecting it entirely, nucleons experience the *same* potential. They are individually spin-half fermions, but with neglect of the difference between neutrons and protons, there is an additional 'isotopic spin' (*isospin*) degeneracy. So the degeneracy of each state becomes $(2I+1) \times (2s+1)$ = 4. For most nuclei there are *many* nucleons, and they fill levels in the nucleus from most tightly bound to most loosely bound, analogous to electrons in an atom. As for all fermion systems, we have two choices:

It is the isospin projection which distinguishes the two, traceable to the quark content of the particle–see Wikipedia.

1. Solve for the energy eigenvalues for protons and neutrons in the potential and occupy levels (including degeneracy) until all nucleons are accounted for. The lowest excitation energy is then the energy between the highest occupied state and the next available empty (discrete energy) state. This is moderately complicated since the Schrödinger equation must be solved [6, 7, 8] for many bound

states: for the central potential $V(r)$ the Schrödinger equation becomes

$$\left[\frac{p_r^2}{2m} + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]u(r) = Eu(r) \qquad (13)$$

where $u(r) = r\psi(\mathbf{r})$. More physical models for $V(r)$ may be found in introductory material on nuclear physics.

2. If the main effect of the potential is to confine particles to a region of space (that is, the nucleon-nucleon interaction is negligible), we have the choice of treating them as a gas of fermions confined to a finite volume. Then the emission of alpha particles come from the highest occupied states, at the 'Fermi level' in the language of fermion statistics. As usual, this proceeds from computing the density of available *nucleon* free-particle states, each of energy $\frac{p^2}{2m}$, where $m$ is the average nucleon mass.

This analysis will be familiar to those with backgrounds in condensed matter physics.

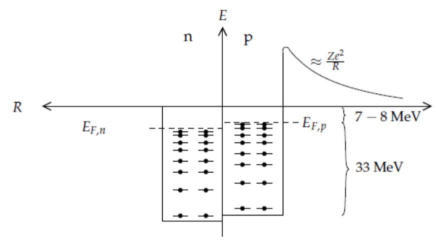


Figure 4: [Unknown source] Schematic energy structure for nucleons, showing both the square well potentials (slightly deeper for neutrons since they experience no Coulomb effects) and Fermi energies for neutrons and protons.

These two viewpoints are presented simultaneously in Fig. 4.

The calculations below are taken from the excellent chapter [2] by the University of Maryland. The book by Demtröder [3] (possibly availablek free to academics) is beautifully illustrated and translated by a non-native English speaker.

$$dN = 4 \times \Omega\frac{d^3p}{h^3} \to 4 \times \Omega\frac{4\pi p^2 dp}{h^3} = 4\frac{\Omega}{(2\pi)^3}k^2 dk \qquad (14)$$

where $\mathbf{p} = \hbar\mathbf{k}$, using the free-particle energy and adopting spherical polar coordinates in momentum $\mathbf{p}$ space. If we integrate over all occupied values of $p$ (by hypothesis, ranging from 0 to $p_F$, the 'Fermi momentum'), we find $N = A$ (=atomic weight) nucleons:

$$A = \frac{2\Omega}{3\pi^2}k_F^3 \to \rho = \frac{A}{\Omega} = \frac{2}{3\pi^2}k_F^3. \qquad (15)$$

where $\rho$ is the number density of nucleons. The kinetic energy of these nucleons $\langle T\rangle$ is

$$\langle T\rangle = \frac{4\Omega}{(2\pi)^3}\left[\int_0^{k_F} dk\, k^2\, \frac{\hbar^2k^2}{2m} = \frac{\hbar^2k_F^5}{5}\right] \qquad (16)$$

$$= \frac{3}{5}E_F, \qquad (17)$$

where $E_F = \frac{\hbar^2k_F^2}{2m}$, eliminating the prefactor using the expression for $N = A$. Evidently the 'Fermi energy' is the energy of the highest occupied nucleon free-particle state. Once again using the estimate $R = 1.2A^{1/3}$ in fermis (fm), the nuclear density and Fermi energies are

$$k_F \simeq 1.433\ \text{fm}^{-1} \qquad (18)$$

$$E_F \simeq 33.4\ \text{MeV}. \qquad (19)$$

Given an average binding energy per nucleon of 7-8 MeV (see Fig. 5 and next section), if we choose to describe the nuclear binding potential as a square well, it must be roughly 33 MeV (the Fermi energy) plus 7-8 MeV (so that nucleons not spontaneously 'spill out' of the well), for a total well depth of 40-41 MeV. It is common for Woods-Saxon potentials to have depths of about 50 MeV.

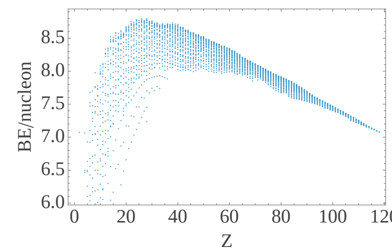


Figure 5: From Mathematica: tabulated binding energy per nucleon

### *Semi-empirical mass formula*

The semi-empirical mass formula[2] is a parametrization of the binding energy per nucleon informed by physical insight into distinct energy contributions for a finite-sized cluster. It has the form

$$BE(Z,N) = a_{vol}A - a_{surf}A^{2/3} - a_{coul}\frac{Z(Z-1)}{A^{1/3}} - a_{asym}\frac{(N-Z)^2}{A} + \Delta E_{pair}, \quad (20)$$

where $A = Z + N$ for $N$ neutrons in a nucleus.

Here

$$\Delta E_{pair} = \sigma\frac{a_{pair}}{\sqrt{A}};\ \sigma = \begin{cases} +1 & \text{even-even} \\ 0 & \text{even/odd, odd/even} \\ -1 & \text{odd-odd} \end{cases}$$

where, for example, odd-odd means N and Z both odd and (with values in MeV),

| symbol | value |
|---|---|
| $a_{vol}$ | 15.85 |
| $a_{surf}$ | 18.34 |
| $a_{coul}$ | 0.71 |
| $a_{asym}$ | 23.21 |
| $a_{pair}$ | 12 |

The Coulomb self-energy of a uniformly charged sphere of charge $Q$ and radius $R$ is $\frac{3Q^2}{5R}$ and $R \propto A^{1/3}$. Here the repulsive proton-proton energy reflects discrete charges.

Terms with negative signs *de-stabilize* the nucleus. Those familiar with finite-size thermodynamics will understand the first two terms, with the second proportional to the number of 'under-coordinated' surface nucleons. The final two terms are peculiar to nuclei and acknowledge, respectively, the preference for $N = Z$ and for protons and neutrons to occur in pairs in the nucleus.

The binding energy (per nucleon) of the *products* of an alpha decay must be lower than the energy of 'reactants' for the decay to be energetically favorable. The kinetic energy $Q$ carried off by an emitted alpha particle is thus $Q = [BE(Z-2, N-2) + BE(2,2)] - BE(Z,N) > 0$

The results for both the WKB half-lives and for alpha kinetic energies after emission are shown in Fig. 6

We have thus satisfactorily answered both of our initial questions.

### *Embellishments*

- In contrast with the stationary description above, it's interesting to ask whether alpha particle tunneling can be treated by time-dependent perturbation theory. In this picture at time $t = 0$ the alpha particle wavefunction is localized within the nucleus and, under the action of the perturbations present, tunnels through the barrier. As usual, time evolution is via the time-dependent Schrödinger equation ,

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = \hat{H}\psi(r,t) \quad (21)$$

$$\Rightarrow \psi(r,t) = e^{\frac{-i\hat{H}t}{\hbar}}\psi(r,t=0) \quad (22)$$

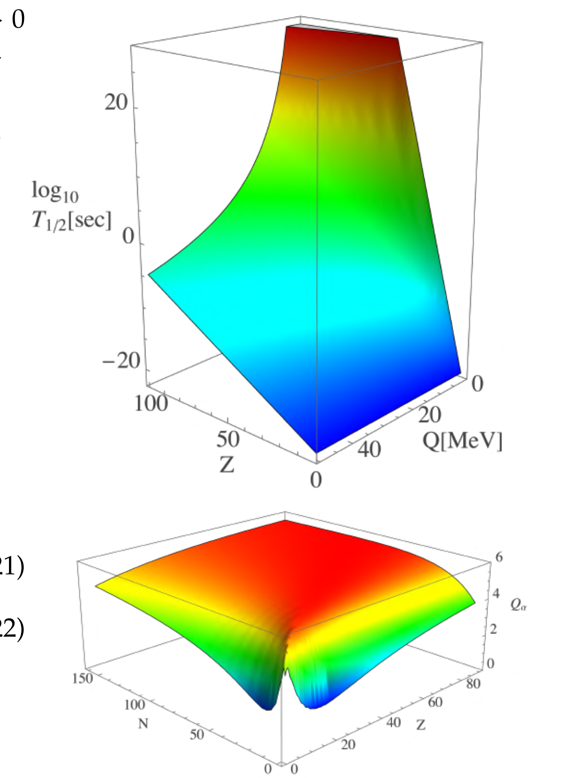


Figure 6: Results of WKB calculatiion for half-lives of alpha emitters, top panel. Lower panel: Semiempirical mass formula results for dependence of emitted alpha kinetic energy $Q$ on $Z$ and $N$.

where in the coordinate representation

$$\hat{H} = -\frac{\hbar^2}{2m_\alpha}\nabla^2 + V(\mathbf{r}) \tag{23}$$

In a basis of bound-state energy eigenstates, details of an implementation with a fairly realistic (Woods-Saxon) nuclear potential (whose form is very reminiscent of the Fermi-Dirac occupation factor) are given in the article by Serot, Carjan, and Strottman [4] and in Nanni [5].

Such TDSE numerical solutions show transient behavior before the Gamow-type exponential decay and details within the nucleus and around the barrier, but recover the WKB treatment above after short transients.

## *References*

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[10] Nazarewicz, Witold and Sobotka, Lee G., *The lessons learned from ephemeral nuclei*, *Physics Today*, February 2025, pp 30-36. Figure 2(a) used with permission.